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Fall Term
Week 13

1st Applications of the Fundamental Theorem of Calculus

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 13.1: Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,\mathrm{d}x = -4\,, \qquad \int_1^5 f(x)\,\mathrm{d}x = 6\,, \qquad \text{and} \qquad \int_1^5 g(x)\,\mathrm{d}x = 8\,.$$

Find

a) $\int_2^2 g(x)\,\mathrm{d}x$ **b)** $\int_2^5 f(x)\,\mathrm{d}x$ **c)** $\int_1^5 (4f(x) - g(x))\,\mathrm{d}x$

Exercise 13.2: Suppose that $\int_1^2 f(x)\,\mathrm{d}x = 5$. Find **a)** $\int_1^2 f(u)\,\mathrm{d}u$ and **b)** $\int_1^2 [-f(x)]\,\mathrm{d}x$

Exercise 13.3: Evaluate $\int_0^2 \left(3x^2 + x - 5\right)\mathrm{d}x$.

Exercise 13.4: Evaluate the following definite integrals: **a)** $\int_0^1 \cos(\frac{1}{2}\pi t)\,\mathrm{d}t$, and **b)** $\int_1^2 \frac{\mathrm{e}^{1/x}}{x^2}\,\mathrm{d}x$.

Exercise 13.5: (Application of Substitution)

a) If f is continuous and $\int_0^4 f(x)\,\mathrm{d}x = 10$, find $\int_0^2 f(2x)\,\mathrm{d}x$.

b) If f is continuous and $\int_0^9 f(x)\,\mathrm{d}x = 4$, find $\int_0^3 x \cdot f(x^2)\,\mathrm{d}x$.

Exercise 13.6: (Integrals and areas between curves.)

a) Set-up an integral for the area of the shaded region given in figure 1 (a) and evaluate this integral to find the area.

b) Set-up an integral for the area of the shaded region given in figure 1 (b) and evaluate this integral to find the area.

c) Set-up an integral for the area of the shaded region given in figure 1 (c) and evaluate this integral to find the area.

Exercise 13.6: Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area A of the region.

a) $y = 1/x$, $y = 1/x^2$, $x = 2$.

b) $x = 1 - y^2$, $x = y^2 - 1$.

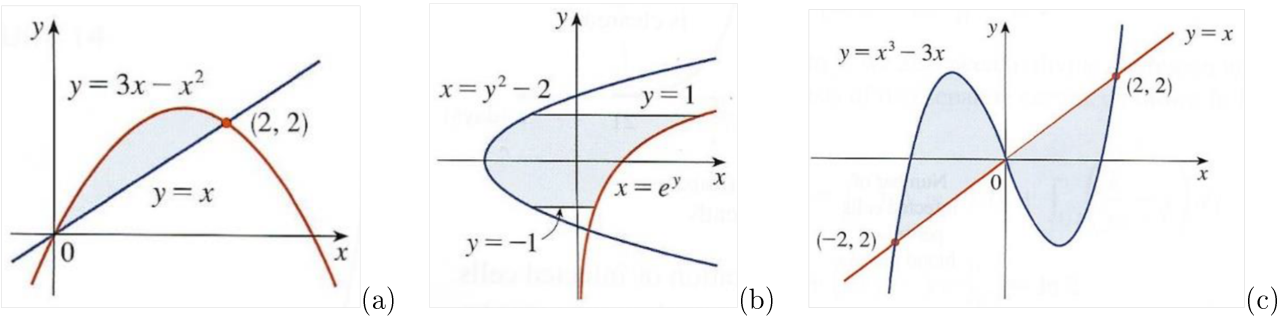

Figure 1: Areas to be computed in exercise 13.6.

Exercise 13.7: Sketch the region enclosed by the given curves and find its area A.

a) $y = \sqrt{x-1}$, $x - y = 1$. **b)** $y = 1/x$, $y = x$, $y = \frac{1}{4}x$, $x > 0$.

Exercise 13.8: Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Exercise 13.9: Which of the following areas are equal? Why?

$$\int_0^1 \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x\,, \qquad \int_0^1 2x\mathrm{e}^x\,\mathrm{d}x\,, \qquad \text{and} \qquad \int_0^{\pi/2} \mathrm{e}^{\sin(x)}\sin(2x)\,\mathrm{d}x\,.$$

Homework Assignment:

Problem 13.1: The Substitution Rule for definite integrals.

a) Evaluate the definite integral by making a suitable substitution.

(i) $\int_0^1 \cos\left(\frac{1}{2}\pi x\right)\,\mathrm{d}x$ **(iii)** $\int_0^1 \sqrt[3]{1+7x}\,\mathrm{d}x$

(ii) $\int_0^1 (3x-1)^{50}\,\mathrm{d}x$ **(iv)** $\int_0^3 \frac{1}{5x+1}\,\mathrm{d}x$

b) Evaluate the definite integral by making a suitable substitution.

(i) $\int_0^1 x\mathrm{e}^{x^2}\,\mathrm{d}x$ **(iii)** $\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,\mathrm{d}x$

(ii) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,\mathrm{d}x$ **(iv)** $\int_{\mathrm{e}}^{\mathrm{e}^4} \frac{1}{x\ln(x)}\,\mathrm{d}x$

c) Evaluate $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x$ by making a substitution and interpreting the resulting integral in terms of an area.

Problem 13.2: The area between curves.

a) Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

(i) $y = \mathrm{e}^x$, $y = x^2 - 1$, $x = -1$, $x = 1$. **(iii)** $y = (x-2)^2$, $y = x$.

(ii) $y = \sin(x)$, $y = x$, $x = \frac{1}{2}\pi$, $x = \pi$. **(iv)** $4x + y^2 = 12$, $x = y$.

b) Sketch the region enclosed by the given curves and find its area.

(i) $y = 12 - x^2$, $y = x^2 - 6$.

(ii) $y = x^2$, $y = 4x - x^2$.

(iii) $y = \cos(x)$, $y = 2 - \cos(x)$, $0 \le x \le 2\pi$.

(iv) $y = x^4$, $y = 2 - |x|$.

c) Use calculus to find the area of the triangle with the given vertices $(0, 0)$, $(3, 1)$, $(1, 2)$.

d) Evaluate the integral $\int_{-1}^{1} |3^x - 2^x| \, \mathrm{d}x$ and interpret it as the area of a region. Sketch the region.

Problem 13.3: Applications in business and economics.

a) The manager of a shoe store determines that the price p USD for each pair of a popular brand of sports sneakers is changing at the rate of when x (hundred) pairs are demanded by consumers. When the price is 75 USD per pair, 400 pairs ($x = 4$) are demanded by consumers.

(i) Find the demand (price) function $p(x)$.

(ii) At what price will 500 pairs of sneakers be demanded? At what price will no sneakers be demanded?

(iii) How many pairs will be demanded at a price of 90 per pair?

b) Suppose that t years from now, one investment plan will be generating profit at the rate of $P_1(t) = 100 + t^2$ thousand GEL per year, while a second investment will be generating profit at the rate of $P_2(t) = 220 + 2t$ thousand GEL per year.

(i) For how many years does the rate of profitability of the second investment exceed that of the first?

(ii) Compute the area enclosed by the two investment plan functions, i.e. the 'net excess profit' (more on this in the next lecture in week 14).

c) Kutaisi Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5000 \left(1 - \frac{100}{(t+10)^2}\right) \qquad [\text{calculators/ week}] .$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

d) After t hours on the job, one factory worker is producing $Q_1(t) = 60 - 2(t-1)^2$ units per hour, while a second worker is producing $Q - 2(t) = 50 - 5t$ units per hour.

(i) If both arrive on the job at 8:00 A.M., how many more units will the first worker have produced by noon than the second worker?

(ii) Interpret the answer in part **(i)** as the area between two curves.