Prof. Dr. Giorgi Chelidze
Prof. Dr. Malkhaz Shashiashvili
Prof. Dr. Dr. h.c. Florian Rupp

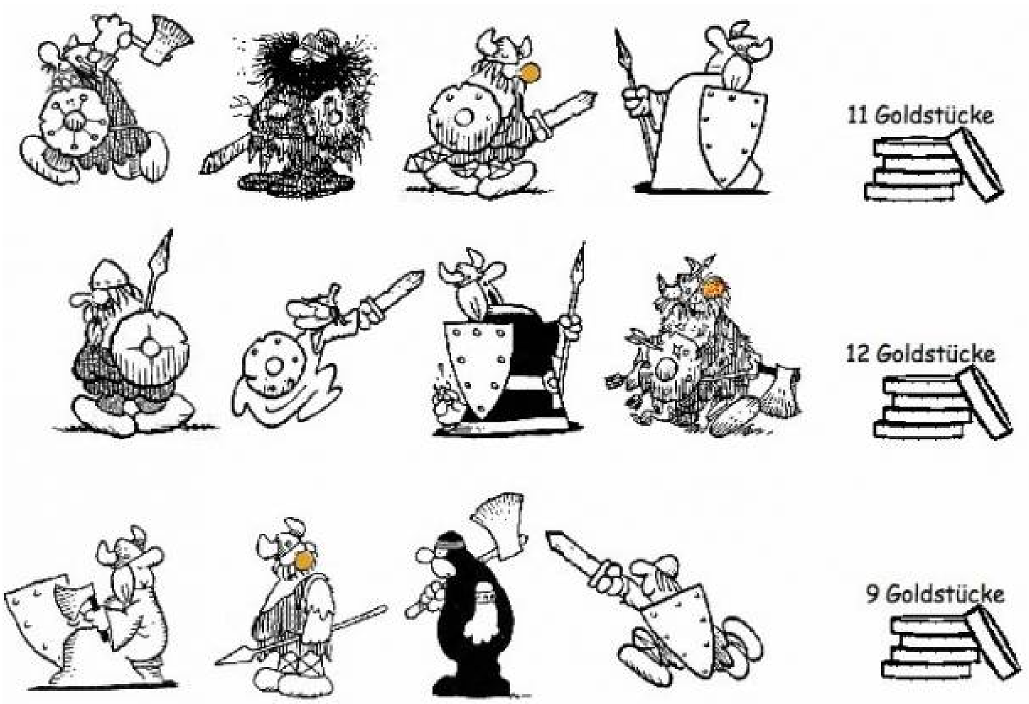
Spring Term
Week 1

Linear Systems of Equations

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 1.1: After returning home from successful plundering Hagar the Horrible remunerates his wild horde of vikings with gold coins (Goldstücke). Though not every branch of service gets the same amount as we can see from his accounting list below (by the way Hagar is the one with the colored nose):[1]



The remuneration is according to the following simple rules:

- There are only 3 branches of service: ax fighter, sword fighter, and lance fighter.
- Each member of a branch of service gets the same amount of gold coins.
- Unfortunately, the third rule got lost over the centuries.

Thus, how the remuneration depends on the branch of service is not so easy to see at a first glance.

[1] All material is subject to the creative commons license CC BY-NC-SA 3.0. This exercise as well as all graphics are taken from the open educational resources at `http://mphinfo.bplaced.net/doku.php` and translated into English and modified to the purposes of our course appropriately (`http://mphinfo.bplaced.net/doku.php?id=mathe:ueben:gausstrainer`).

Though, we can use linear systems to make good for that.

a) [Finding the 3rd Rule] Set-up a suitable system of linear equations to obtain the third rule and to determine how the remuneration depends on the branch of service exactly.

b [But Hagar is the Boss] Since decades historians debate about the actual money Hagar has taken. Some think that he pays himself always double the amount that the others from the same branch of service get.

- How would the calculation look subject to these conditions?
- Why can this assumption not be true just like that?
- Change the total remuneration in the third line such that the contradiction vanishes, and compute then the income of each member of a branch of service.

Exercise 1.2: Depending on the choice of the real constants a and b give the feasible solution set of the following system of two equations in two unknowns. I.e., determine $a, b \in \mathbb{R}$ such that this system may have none, exactly one, or infinitely many solutions.

a) $x + by = -1$ and $ax + 2y = 5$.

b) $x - 2y = 1$ and $ax + by = 5$

Exercise 1.3: Determine the solution set of the following linear systems of equations by bringing them first to upper echelon form (for the augmented matrix) and then perform a back-substitution to gain the results (make sure you write down the elementary operations you use). Finally, interpret the result geometrically:

a) $x_1 + 5x_2 = 7$ and $-2x_1 - 7x_2 = -5$.

b) $x_1 + x_2 - x_3 = 1$, $2x_1 + x_2 - x_3 = 6$ and $3x_1 + 7x_2 - 7x_3 = -13$.

Homework Assignment:

Problem 1.1: Reviewing Calculus I.

a) Evaluate the definite integrals

$$\int_0^1 \left(\mathrm{e}^{2x} + 4\sqrt[3]{x}\right) \mathrm{d}x \qquad \text{and} \qquad \int_0^1 \mathrm{e}^{-x} \left(\mathrm{e}^{-x} + 1\right)^{1/2} \mathrm{d}x\,.$$

b) Find the function whose tangent line has slope $(x+1)\mathrm{e}^{-x}$ for each value of x and whose graph passes through the point $(1, 5)$.

c) A manufacturer has determined that when q thousand units of a particular commodity are produced, the price at which all the units can be sold is $p = D(q)$ GEL per unit, where D is the demand function

$$D(q) \;=\; 10 - q\mathrm{e}^{0.02 \cdot q}\,.$$

At what price are 5000 ($q_0 = 5$) units demanded? What is the value of the consumers' surplus when 5000 units are demanded.

d) A manufacturer determines that when x hundred units of a particular commodity are produced, the profit generated is $P(x)$ thousand GEL, where

$$P(x) \;=\; \frac{500 \cdot \ln(x+1)}{(x+1)^2}$$

What is the average profit over the production range $0 \le x \le 10$?

Problem 1.2: Determine the solution set of the following linear systems of equations by bringing them first to upper echelon form (for the augmented matrix) and then perform a back-substitution to gain the results (make sure you write down the elementary operations you use). Finally, check your result and interpret the result geometrically:

a) $x_1 - 2x_2 - 6x_3 = 12$, $2x_1 + 4x_2 + 12x_3 = -17$ and $x_1 - 4x_2 - 12x_3 = 22$.

b) $x_1 - x_2 + 3x_3 = 5$, $-x_1 + x_2 + x_3 = -1$ and $3x_1 - 2x_2 + x_3 = -2$.

c) $x_1 + x_2 - x_3 = 1$, $2x_1 + x_2 - x_3 = 6$ and $3x_1 + 7x_2 - 6x_3 = -1$.

d) $x_1 + x_2 - x_3 = 1$, $2x_1 + x_2 - x_3 = 6$ and $3x_1 + 7x_2 - 7x_3 = 1$.

Problem 1.3: National income models express the equilibrium level of income generally as

$$Y \;=\; C + I + G + (X - Z)\,.$$

Where Y is the aggregate income generated in the economy from aggregate consumption C, investment I, government expenditure G, and net export (export X minus import Z), generally measured in a year. Alternatively, Y is the total income and $C + I + G + (X - Z)$ is the total expenditure in the economy in a year and the equilibrium national income occurs when the total income and total expenditure are equal.

Assume a simple three-sector economy where

$$Y - C - I \;=\; 20\,, \qquad C - 0.7Y \;=\; 65\,, \qquad \text{and} \qquad I - 0.1Y \;=\; 50\,.$$

Find the equilibrium income Y and equilibrium consumption C.