Kutaisi International University — Calculus II for Management

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Determinants & Linear Combinations of Vectors

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 3.1: Compute the values of the following determinant

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{pmatrix} .$$

Exercise 3.2: Compute the determinant

$$\det \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Hint: Long calculations are not required, focus on the structure and the properties of the determinant.

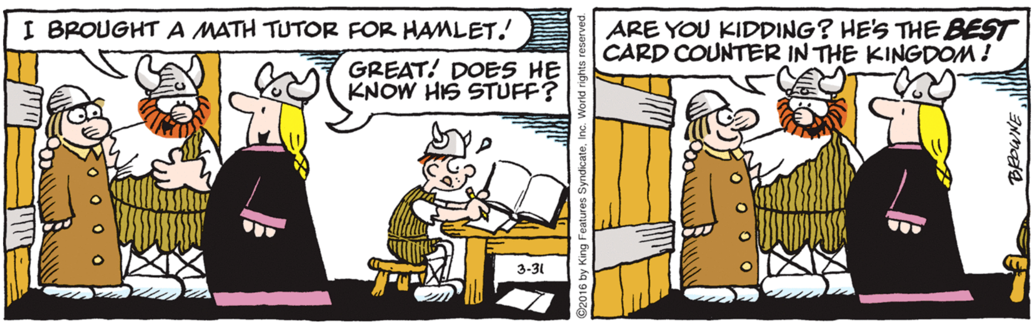
Exercise 3.3: Decide whether the following vectors are linearly independent or not:

a) $\vec{v}_1 = (1, 2)^T$ and $\vec{v}_2 = (2, 1)^T$

b) $\vec{v}_1 = (1, 2, 1, 2)^T$, $\vec{v}_2 = (-1, 0, 0, 2)^T$ and $\vec{v}_3 = (1, -1, 0, 0)^T$

Exercise 3.4: The vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are a basis of $\mathbb{R}^3$. Find the unique coordinate triple $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ to represent $\vec{v}_4$ as a linear combination of the basis vectors, i.e. $\vec{v}_4 = \sum_{i=1}^{3} \lambda_i \vec{v}_i$. Consider

$$\vec{v}_1 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix}, \quad \text{and} \quad \vec{v}_4 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} .$$



Homework Assignment:

Problem 3.1: Compute the determinants

a) $\det \begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 1 & 5 & 4 & 0 & -9 \\ 3 & 1 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 & 3 \\ 1 & 0 & 3 & 0 & 1 \end{pmatrix}$. b) $\det \begin{pmatrix} 1 & 1 & 1 & -1 & 2 & -4 \\ -4 & -1 & 2 & 2 & 3 & 9 \\ 1 & 1 & 1 & -1 & 2 & -4 \\ 7 & 9 & 8 & 11 & 0 & 3 \\ 23 & 19 & 0 & 19 & 4 & 49 \\ 99 & -8 & 4 & -7 & 5 & 81 \end{pmatrix}$.

Hint: Long calculations are not required, focus on the structure and the properties of the determinant.

Problem 3.2: Let V be the set of all elements in $\mathbb{R}^3$ with third entry equal to 0, i.e.,

$$V := \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3 \, : \, x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3 .$$

Then V is a vector space with the usual addition and scalar multiplication.

To verify this, we need only check that V is closed under addition and scalar multiplication. Associativity and all the other required properties hold because they hold for $\mathbb{R}^3$ (and hence for this subset of $\mathbb{R}^3$). Furthermore, if we can show that V is closed under scalar multiplication and addition, then for any particular $\vec{v} \in V$, $0 \cdot \vec{v} = \vec{0} \in V$. So we simply need to check that $V \neq \{\}$, that if $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$, and if $\alpha \in \mathbb{R}$ and $\vec{v} \in V$ then $\alpha \cdot \vec{v} \in V$.

Task: Verify that for $\vec{u}, \vec{v} \in V$ and $\alpha \in \mathbb{R}$, $\vec{u} + \vec{v} \in V$ and $\alpha \cdot \vec{v} \in V$.

Problem 3.3: Decide whether the following vectors are linearly independent or not:

a) Vectors in $\mathbb{R}^2$:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 928094 \\ 257567 \end{pmatrix}, \begin{pmatrix} 234234 \\ 23442 \end{pmatrix}, \begin{pmatrix} 192 \\ 293849 \end{pmatrix}.$$

b) Vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -a \\ b \\ -c \end{pmatrix}.$$

In particular at the last example, how is linear dependance and independence connected to the values of $a, b, c \in \mathbb{R}$?

Problem 3.4: Compute a vector that is orthogonal to.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix},$$

and determine the value of the Cosine of the angle between $\vec{v}_1$ and $\vec{v}_2$.

Problem 3.5: Compute the following products of matrices wherever possible. In the cases where you can not multiply the matrices, justify your decision.

a) $\begin{pmatrix} 0 & 2 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} 6 & -6 \\ 3 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 10 & 2 & -7 & -2 \\ 4 & 9 & 0 & -8 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & -1 \\ -9 & 5 \\ 8 & -4 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & -5 \\ 5 & -1 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 8 & -6 & -2 \\ 9 & -7 & -3 \\ 2 & 0 & 10 \end{pmatrix} \begin{pmatrix} -3 & -8 & 10 \\ 5 & -3 & -5 \\ 9 & -1 & 8 \end{pmatrix}$