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Week 4

Vector & Matrix Computations

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 4.1: Find a vector $\vec{v} \in \mathbb{R}^3$ such that the following three vectors form a basis of $\mathbb{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} -2 \\ 2 \\ -5 \end{pmatrix}, \qquad \text{and} \qquad \vec{v},$$

and compute the value of the Cosine of the angle between $\vec{u}_1$ and $\vec{u}_2$.

Exercise 4.2: Compute the values of the following products if they exist:

$$\left\langle \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\rangle, \quad \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & 5 \\ -2 & 2 \end{pmatrix}$$

Exercise 4.3: Professor Dumbledore from the university of Hogwarts writes his office and home phone number as a 5×1-matrix O and as an 1×5-matrix H, respectively. As professor for transfiguration he enjoys any possibility to rearrange and transform objects. Help him compute $\det(OH)$.

Homework Assignment:

Problem 4.1: Compute a vector that is orthogonal to.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix},$$

and determine the value of the Cosine of the angle between $\vec{v}_1$ and $\vec{v}_2$.

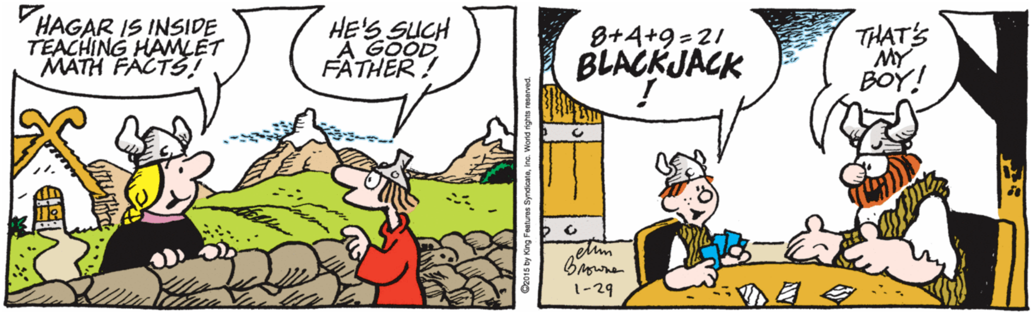
Problem 4.2: Compute the following products of matrices wherever possible. In the cases where you can not multiply the matrices, justify your decision.

a) $\begin{pmatrix} 0 & 2 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} 6 & -6 \\ 3 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 10 & 2 & -7 & -2 \\ 4 & 9 & 0 & -8 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & -1 \\ -9 & 5 \\ 8 & -4 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & -5 \\ 5 & -1 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 8 & -6 & -2 \\ 9 & -7 & -3 \\ 2 & 0 & 10 \end{pmatrix} \begin{pmatrix} -3 & -8 & 10 \\ 5 & -3 & -5 \\ 9 & -1 & 8 \end{pmatrix}$



Problem 4.3: Leontief Input-Output Models. Suppose we have 2 factories and consumers. There is consumer demand for the products of the factories. In addition, each factory needs products from their own factory and the other factories in order to produce their product. We would like to find the output level for each factory so that all the demands are met with no product left over. Let $\vec{d} = (d_1, d_2)^T$ be the consumer demand vector for the products. Let $\vec{x} = (x_1, x_2)$ be the output vector of the two factories. We measure the demand and output in terms of units (which could be GEL). Let $a_{i,j}$ be the number of units that factory i sends to factory j for each unit that factory j produces. Let $A = (a_{i,j})$ be a 2×2-matrix.

This economic situation leads to the following system of equations

$$\begin{aligned} x_1 &= a_{1,1}x_1 + a_{1,2}x_2 + d_1 \\ x_2 &= a_{2,1}x_1 + a_{2,1}x_2 + d_2 \end{aligned}$$

that can be written as

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

or

$$(\mathbb{I} - A)\,\vec{x} = \vec{d},$$

where $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2-unit matrix, and $(\mathbb{I} - A)$ is called the **Leontief matrix**.

Suppose, factory 1 (a power plant) produces electricity and factory 2 (a municipal water facility) produces water. To have output from the two factories measured in common units, we measure both outputs in GEL. For each unit of electricity produced, factory 1 must use 0.2 units of electricity and 0.4 units of water. For each unit of water produced, factory 2 must use 0.3 units of electricity and 0.1 units of water. Also, consumer demand is $d_1 = 10$ units of electricity and $d_2 30$ units of water. Thus, we have

$$A = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}.$$

Question: Compute the output vector $\vec{x} = (x_1, x_2)$ for these two factories so that the demands of both the facilities and the consumers are satisfied.