Prof. Dr. Giorgi Chelidze
Prof. Dr. Malkhaz Shashiashvili
Prof. Dr. Dr. h.c. Florian Rupp

Spring Term
Week 6

Improper Integrals

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 6.1: Find the characteristic polynomial, all eigenvalues, and representative eigenvectors corresponding to these eigenvalues of the following matrices

$$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}.$$

Exercise 6.2: For what values of p is the following integral convergent

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x\,.$$

Exercise 6.3: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}\,\mathrm{d}x\,, \qquad \int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,\mathrm{d}x\,, \qquad \text{and} \qquad \int_{-\infty}^0 \frac{1}{3-4x}\,\mathrm{d}x\,.$$

Exercise 6.4: Evaluate $\int_0^1 \ln(x)\,\mathrm{d}x$ (improper integral of type 2).

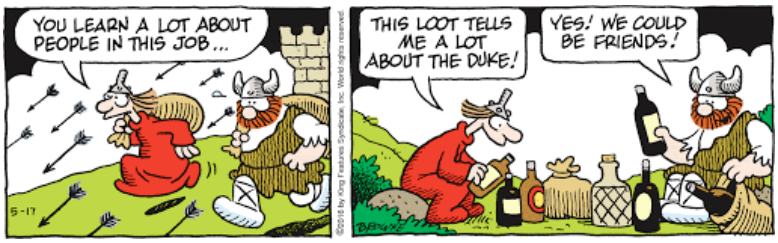
Exercise 6.5: In each case, use an improper integral to estimate the total population TP of the urban area, where the population density $p(r)$ of the urban area is given as

a) $\rho(r) = 100 \cdot \mathrm{e}^{-0.02\,r}$, and
b) $\rho(r) = 100 \cdot \mathrm{e}^{-0.02\,r} + 2000 \cdot \mathrm{e}^{-0.001\,r^2}$.

Homework Assignment:

Problem 6.1: Find the characteristic polynomial, all eigenvalues (if they exist), and representative eigenvectors corresponding to these eigenvalues of the following matrices

$$\begin{pmatrix} 7 & -2 \\ 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & -1 \\ -1 & -7 & -5 \\ 1 & 9 & 7 \end{pmatrix}.$$



Problem 6.2: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x$ b) $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,\mathrm{d}x$ c) $\displaystyle\int_{-\infty}^\infty x\cdot \mathrm{e}^{-x^2}\,\mathrm{d}x$ d) $\displaystyle\int_2^\infty \mathrm{e}^{-5x}\,\mathrm{d}x$

Problem 6.3: The capitalized cost of an asset is the sum of the original cost of the asset and the present value of maintaining the asset. Suppose a company is considering the purchase of two different machines. Machine 1 costs 40000 GEL and t years from now will cost $M_1(t) = 4000(1 + 0.06t)$ GEL per year to maintain. Machine 2 costs only 32000 GEL, but its maintenance cost at time t is $M_2(t) = 4400$ GEL per year.

a) If the cost of money is 9% per year compounded continuously, what is the capitalized cost of each machine? Which machine should the company purchase?

b) Research various methods used by economists to make comparisons between competing assets. Write a paragraph comparing these methods.

Problem 6.4: The manager of an electronics firm estimates that the proportion of components that last longer than t months is given by the improper integral

$$\int_t^\infty 0.008\,\mathrm{e}^{-0.008\,x}\,\mathrm{d}x\,.$$

Which is larger, the proportion of components that last longer than 5 years (60 months) or the proportion that fail in less than 10 years?

Problem 6.5: Find the accumulated present value of an investment for which there is a perpetual continuous money flow of $2000\mathrm{e}^{-0.01t}$ GEL per year, assuming continuously compounded interest at a rate of 7%.

Problem 6.6: A company determines that its marginal cost, in dollars, for producing x units of a product is given by

$$C'(x) \;=\; 3600x^{-1.8}\,, \qquad \text{where } x \geq 1\,.$$

Suppose that it were possible for the company to make infinitely many units of this product. What would the total cost be?

Problem 6.7: The capitalized cost c of an asset for an unlimited lifetime is the total of the initial cost and the present value of all maintenance expenses that will occur in the future. It is computed by the formula

$$c \;=\; c_0 + \int_0^\infty m(t)\mathrm{e}^{-rt}\,\mathrm{d}t\,,$$

where c_0 is the initial cost of the asset r is the interest rate (compounded continuously), and $m(t)$ is the annual cost of maintenance. Find the capitalized cost under each set of assumptions.

a) $c_0 = 500000$ GEL, $r = 5\%$, $m(t) = 20000$ GEL.

b) $c_0 = 700000$ GEL, $r = 5\%$, $m(t) = 30000$ GEL.