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Spring Term
Week 7

Numeric Integration & Inverse Matrices

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 7.1: Use finite approximations to estimate the area under the graph of the function

$$f(x) \;=\; \frac{1}{x} \qquad \text{between } x = 1 \text{ and } x = 5$$

using a lower sum with rectangles of equal width.

a) first two rectangles and
b) second four rectangles,

and using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (the midpoint rule), using

c) first two rectangles and
d) second four rectangles.

Exercise 7.2: Given

$$f(x) \;=\; x^2 + 1 \qquad \text{over the interval } [0, 3]\,.$$

a) Find a formula for the Riemann sum obtained by dividing the interval $[0, 3]$ into n equal subintervals and using the right-hand endpoint for each c_k.
b) Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0, 3]$.

Exercise 7.3: An economist studying the demand for a particular commodity gathers the data in the accompanying table, which lists the number of units q (in thousands) of the commodity that will be demanded (sold) at a price of p GEL per unit.

q (1000 units)	0	4	8	12	16	20	24
p (GEL/ unit)	49.12	42.90	31.32	19.83	13.89	10.58	7.25

Use this information together with Simpson's rule to estimate the total revenue

$$R \;=\; \int_0^{24} p(q)\,\mathrm{d}q \qquad \text{thousand GEL}$$

obtained as the level of production is increased from 0 to 24000 units ($q = 0$ to $q = 24$).

Exercise 7.4: Given

$$\vec{v}_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \quad \text{and} \quad \vec{v}_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

as well as

$$A \;=\; \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

a) Find the values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3 = \vec{0}$

b) Is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis of $\mathbb{R}$. Justify your answer.

c) Compute the value of $\det(A)$.

d) Determine A^{-1} and check your result by computing AA^{-1}.

e) Find all eigenvalues and their corresponding eigenvectors of A.

Exercise 7.5: Inverting matrices:

a) Is the following matrix invertible and if so compute its inverse (check your result)?

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

b) Is the following matrix invertible and if so compute its inverse (check your result)?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Homework Assignment:

Problem 7.1: Use $n = 6$ subintervals to approximate the integral

$$\int_0^3 \sqrt{9 - x^2}\,\mathrm{d}x$$

with the

a) mid-point rule,
b) trapezoidal rule, and
c) Simpson's rule.

Problem 7.2: On the first day of each month, Hagar estimates the rate at which profit is expected to increase during that month's plundering efforts. The results are listed in the accompanying table for the first 6 months of the year, where $P'(t)$ is the rate of profit growth in thousands of gold coins per month expected during the tth month ($t = 1$ for January, $t = 6$ for June).

t (months)	1	2	3	4	5	6
rate of profit $P'(t)$	0.65	0.43	0.72	0.81	1.02	0.97

Use this information together with the trapezoidal rule to estimate the net profit earned by Hagar and his team during this 6-month period (January through June).

Problem 7.3: Find the rank of the matrix below for all possible values of the parameter α and state for which values of α they are invertible (no need to actually compute the inverse matrix if it exists).

$$\begin{pmatrix} 1 & \alpha & -1 & 2 \\ 2 & -1 & \alpha & 5 \\ 1 & 10 & 6 & 1 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 & 1 & -1 \\ \alpha & 0 & -1 & 0 \\ 0 & 0 & 2\alpha & 2 \\ 1 & 2 & 2 & 3 \end{pmatrix}.$$

Problem 7.4: Given the following matrices. Show whether they are invertible and if so compute their inverse (check your result)?

$$\begin{pmatrix} 5 & 9 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 5 & 5 \\ 1 & 6 & 5 \\ -2 & 7 & 9 \end{pmatrix}.$$