Prof. Dr. Giorgi Chelidze
Prof. Dr. Malkhaz Shashiashvili
Prof. Dr. Dr. h.c. Florian Rupp

Introduction to Continuous Probability

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 8.1: Let $f(x) = 0.006 \cdot x \cdot (1-x)$ for $0 \le x \le 10$ and $f(x) = 0$ for all other values of x. Verify that f is a probability density function and find $\mathbb{P}(4 \le X \le 8)$.

Exercise 8.2: Either find a number k such that the given function is a probability density function or explain why no such number exists:

$$f(x) \;=\; \begin{cases} k - 3x\,, & \text{for } 0 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Exercise 8.3: Find the expected value of the general exponential distribution

$$f(t) \;=\; \begin{cases} 0\,, & \text{if } t < 0 \\ c\,\mathrm{e}^{-ct}\,, & \text{if } t \ge 0 \end{cases}$$

Exercise 8.4: Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.

b) Find the probability that a customer waits more than five minutes to be answered.

Exercise 8.5: Find the expected value for the random variable with the density function given by

$$f(x) \;=\; \begin{cases} \frac{1}{3}\,, & \text{for } 2 \le x \le 5 \\ 0\,, & \text{otherwise} \end{cases}$$

Exercise 8.6: (Product Reliability) The life span of a particular brand of food processor is measured by the random variable X with probability density function

$$f(x) \;=\; \begin{cases} 0.002\,x\,\mathrm{e}^{-0.001\,x^2} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

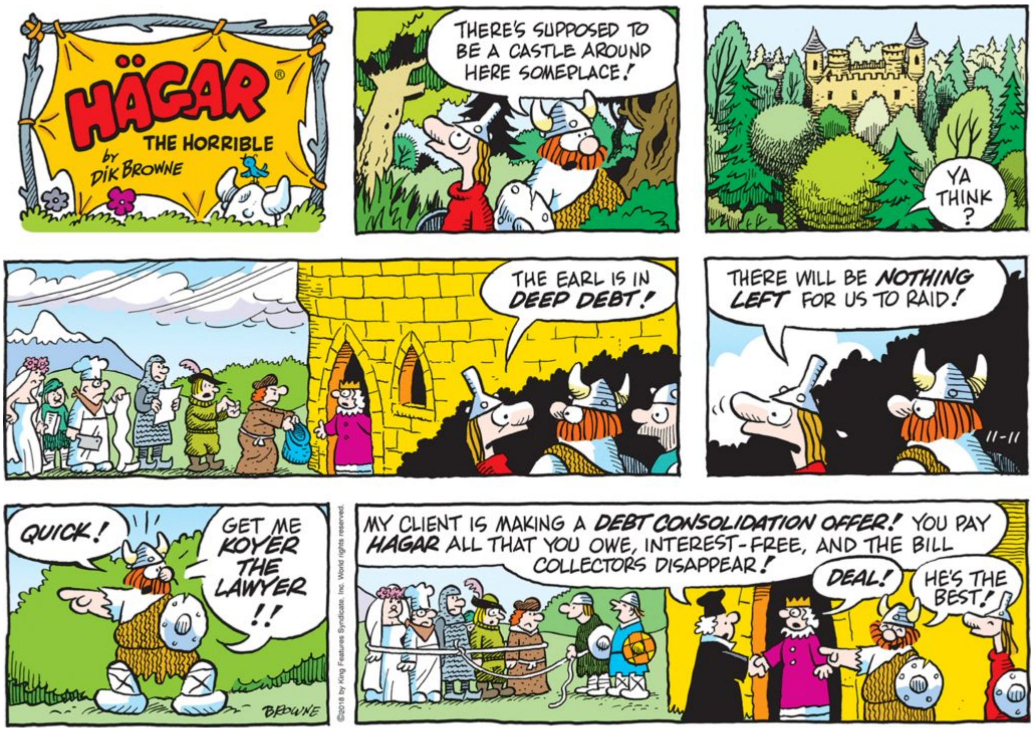
where x denotes the life span (in months) of a randomly selected processor.

a) The expected value of X is given by the improper integral

$$\mathbb{E}(X) \;=\; \int_0^\infty 0.002\,x^2\,\mathrm{e}^{-0.001\,x^2}\,\mathrm{d}x$$

which can be evaluated only by numerical methods. Estimate $\mathbb{E}(X)$ by applying Simpson's rule with $n = 10$. Based on your result, how long would you expect a randomly chosen processor to last?

b) The actual expected value $\mathbb{E}(X)$ is roughly 28 (months). Discuss what (if anything) you could do to improve the result you obtained in part a).



Homework Assignment:

Problem 8.1: Determine whether the given function is a probability density function

$$f_1(x) = \begin{cases} \frac{1}{10}, & \text{for } 30 \le x \le 40 \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad f_2(x) = \begin{cases} \frac{1}{9}\sqrt{x}, & \text{for } 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}.$$

Problem 8.2: Either find a number k such that the given function is a probability density function or explain why no such number exists:

$$f_1(x) = \begin{cases} x\,\mathrm{e}^{-k\,x}, & \text{for } x \ge 0 \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad f_2(x) = \begin{cases} x^3 + kx, & \text{for } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Problem 8.3: $f(x)$ is a probability density function for a particular random variable X. Use integration to find the indicated probabilities.

a) Determine $\mathbb{P}(0 \le X \le 4)$, $\mathbb{P}(2 \le X \le 3)$, and $\mathbb{P}(X \ge 1)$, where

$$f(x) = \begin{cases} \frac{1}{8}(4 - x), & \text{for } 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

b) Determine $\mathbb{P}(X \ge 0)$, $\mathbb{P}(1 \le X \le 2)$, and $\mathbb{P}(X \le 2)$, where

$$f(x) = \begin{cases} 2x\mathrm{e}^{-x^2}, & \text{for } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 8.4: Suppose the time X a customer must spend waiting in line at a certain bank is a random variable that is exponentially distributed with density function

$$f(x) = \begin{cases} \frac{1}{4}\,\mathrm{e}^{-x/4}, & \text{for } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

where x is the number of minutes a randomly selected customer spends waiting in line.

a) Find the probability that a customer will have to stand in line at least 8 minutes.

b) Find the probability that a customer will have to stand in line between 1 and 5 minutes.

c) Find the expected waiting time for customers at the bank.

Problem 8.5: Applications in Business and Economics

a) Assuming an exponentially distributed car density on highways, a transportation planner determines that the average distance between cars on a certain highway is 100 meters. What is the probability that the distance between two successive cars, chosen at random, is 40 meters or less?

b) The **time to failure** t in hours, of a machine is often exponentially distributed with a probability density function

$$f(t) \;=\; k \cdot \mathrm{e}^{-kt}\,, \qquad 0 \le t < \infty\,,$$

where $k = 1/a$ and a is the average amount of time that will pass before a failure occurs. Suppose that the average amount of time that will pass before a failure occurs is 100 hours. What is the probability that a failure will occur in 50 hours or less?

c) The **reliability of the machine** (the probability that it will work) in part b) is defined as

$$R(T) \;=\; 1 - \int_0^T 0.01\mathrm{e}^{-0.01t}\,\mathrm{d}t\,,$$

where $R(T)$ is the reliability at time T. Write without using an integral.

d) Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

(i) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.

(ii) Find the probability that a customer waits more than five minutes to be answered.

(iii) Show that the median waiting time for a phone call to the company is about 3.5 minutes.