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Spring Term
Week 9

Functions in Several Variables

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

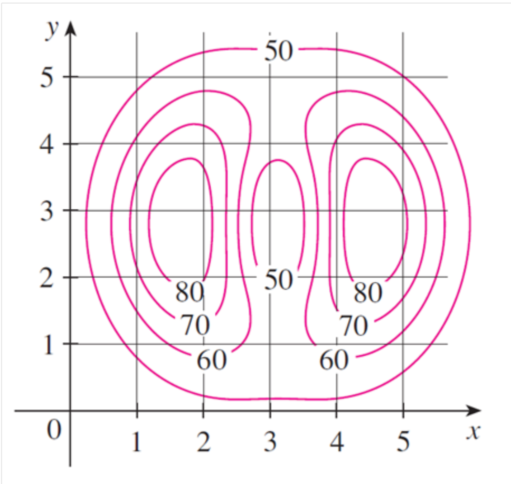
Exercise 9.1: Find the partial derivatives f_x and f_y of $f(x,y) = x\,\mathrm{e}^{-2\,x\,y}$.

Exercise 9.2: Evaluate $f(-1,2,5)$, $f_x(-1,2,5)$, $f_y(-1,2,5)$, and $f_z(-1,2,5)$ for the following function of three variables $f(x,y,z) = x\,y + x\,z + y\,z$.

Exercise 9.3: Find the partial derivatives f_x and f_y for

$$f(x,y) \;=\; \sin\left(\frac{1}{1+y}\right)$$

Exercise 9.4: A contour map for a function f is shown in the figure below. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Exercise 9.5: Sketch some level curves of the function $h(x,y) = 4x^2 + y^2 + 1$.

Exercise 9.6: Consider the graphs and level curve plots given in figure 1. Which graph belongs to which level curve plot?

Exercise 9.7: Find the equation of the tangential plane for the given function at the given point.

a) $f(x,y) = 1 + x \cdot \ln(xy - 5)$ at $(a,b) = (2,3)$.

b) $f(x,y) = \mathrm{e}^x \cdot \cos(xy)$ at $(a,b) = (0,0)$.

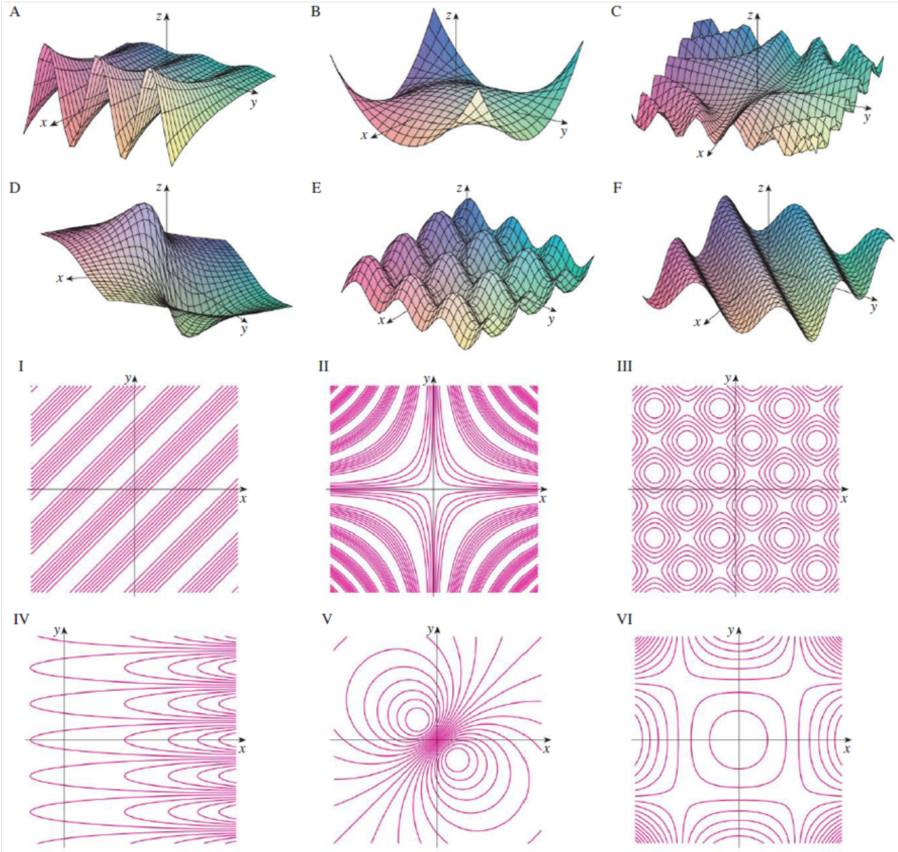


Figure 1: Match the graphs and the level set curves.

Homework Assignment:

Problem 9.1: A sum of 1000 GEL is deposited in a savings account for which interest is compounded monthly. The future value A is a function of the annual percentage rate r and the term t, in months, and is given by

$$A(r,t) \;=\; 1000\left(1+\frac{r}{12}\right)^{12t}.$$

a) Determine $A(0.05, 10)$.

b) What is the interest earned for the rate and term in part a)?

c) How much more interest can be earned over the same term as in part a) if the annual percentage rate r is increased to 5.75%?

Problem 9.2: Find the indicated partial derivative.

a) $f(x,y) = x^4 + 5xy^3$ and $f_x(1,1)$.

b) $f(x,y) = x\,\sin(xy)$ and $f_x(1,\pi)$.

c) $f(x,y,z) = \ln\left(\frac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}\right)$ and $f_y(1,2,2)$.

d) $f(x,y,z) = x^{y\,z}$ and $f_z(\mathrm{e},1,0)$.

Problem 9.3: You are told that there is a function f whose partial derivatives are $f_x(x,y) = x + 4y$ and $f_y(x,y) = 3x - y$. Should you believe it?

Problem 9.4: Find the gradient $\nabla f(x,y)$ and Hessian matrix $H_f(x,y)$ of the given functions.

a) $f(x,y) = 2x - 3y$.

b) $f(x,y) = \mathrm{e}^{2xy}$.

c) $f(x,y) = x + \mathrm{e}^{y}$.

d) $f(x,y) = x \cdot \ln(y)$.

Problem 9.5: Find the equation of the tangential plane for the given function at the given point.

a) $f(x,y) = \sqrt{x \cdot y}$ at $(a,b) = (1,4)$.

b) $f(x,y) = x^2 \cdot \mathrm{e}^{y}$ at $(a,b) = (1,0)$.

c) $f(x,y) = y + \sin(x/y)$ at $(a,b) = (0,3)$.

d) $f(x,y) = \frac{y-1}{x+1}$ at $(a,b) = (0,0)$.

Problem 9.6: Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2,1,3)$. You don't have an equation for S but you know that the curves

$$\left\{ \begin{pmatrix} 2+3t \\ 1-t^3 \\ 3-4t+t^2 \end{pmatrix} : t \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1+s^2 \\ 2s^3-1 \\ 2s+1 \end{pmatrix} : s \in \mathbb{R} \right\}$$

both lie on S. Find an equation of the tangent plane at P.

Problem 9.7: Lincolnville Sporting Goods has the following **Cobb-Douglas production function** for a certain product:

$$p(x,y) = 2400 x^{2/5} y^{3/5},$$

where p is the number of units produced with x units of labor and y units of capital.

a) Find the number of units produced with 32 units of labor and 1024 units of capital.

b) Find the marginal productivities.

c) Evaluate the marginal productivities at $x = 32$ and $y = 1024$.

d) Interpret the meanings of the marginal productivities found in part c).

Problem 9.8: A study of Texas nursing homes found that the annual profit P (in dollars) of profit-seeking, independent nursing homes in urban locations is modeled by the function

$$P(w, r, s, t) = 0.007955 w^{-0.638} r^{1.038} s^{0.873} t^{2.468} .$$

In this function, w is the average hourly wage of nurses and aides (in dollars), r is the occupancy rate (as a percentage), s is the total square footage of the facility, and t is the Texas Index of Level of Effort (TILE), a number between 1 and 11 that measures state Medicaid reimbursement.[1]

a) A profit-seeking, independent Texas nursing home in an urban setting has nurses and aides with an average hourly wage of 20 USD an hour, a TILE of 8, an occupancy rate of 70%, and 400000 square feet of space.

   (i) Estimate the nursing home's annual profit.

   (ii) Find the four partial derivatives of P.

   (iii) Interpret the meaning of the partial derivatives found in part (ii).

b) The change in P due to a change in w when the other variables are held constant is approximately

$$\Delta P \approx \frac{\partial P}{\partial w} \Delta w .$$

Use the values of w, r, s, and t in part a) and assume that the nursing home gives its nurses and aides a small raise so that the average hourly wage is now 20.25 USD an hour. By approximately how much does the profit change?

[1] Source: K. J. Knox, E. C. Blankmeyer, and J. R. Stutzman (1999): Relative Economic Efficiency in Texas Nursing Facilities, Journal of Economics and Finance, Vol. 23, 199-213.