

Partial Derivatives and Optimizing Functions of Two Variables

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 10.1: If $z = x^2y + 3xy^4$, where $x = \sin(2t)$ and $y = \cos(t)$, find $\frac{dz}{dt}$ when $t = 0$.

Exercise 10.2: Find all critical points for the given function, and classify each as a relative maximum, relative minimum, or saddle point.

a) $f(x, y) = x^4 + y^4 - 4xy + 1$

b) $f(x, y) = x^3 - y^3 + 6xy$

Exercise 10.3: Giorgi manages a grocery store that carries two brands of cat food, a local brand obtained at the cost of 30 cents per can and a well-known national brand obtained for 40 cents per can. He estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, then approximately $70 - 5x + 4y$ cans of the local brand and $80 + 6x - 7y$ of the national brand will be sold each day. How should Giorgi price each brand to maximize total daily profit from the sale of cat food? (Assume that the largest daily profit occurs at a relative maximum.)

Exercise 10.4: A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume V of such a box.

Homework Assignment:

Problem 10.1: Find all critical points of the given functions and classify each as a relative maximum, a relative minimum, or a saddle point.

a) $f(x, y) = 5 - x^2 - y^2$.

c) $f(x, y) = (x - 4) \cdot \ln(xy)$.

b) $f(x, y) = x^3 + y^2 - 6xy + 9x + 5y + 2$.

d) $f(x, y) = xy e^{(16x^2 + 9y^2)/288}$.

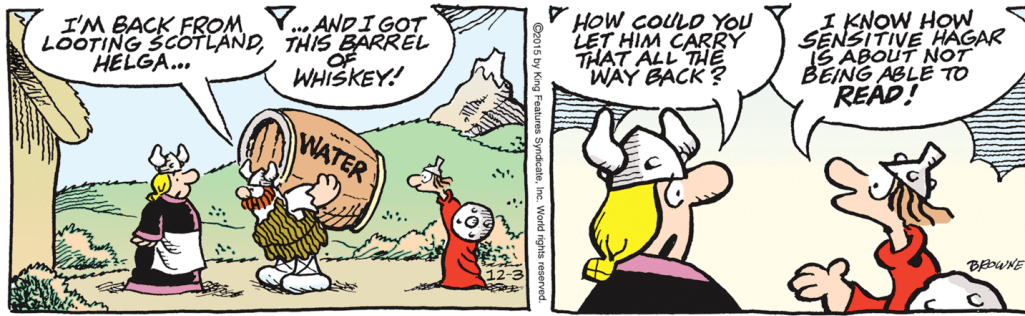
Problem 10.2: A concert promoter produces two kinds of souvenir shirt; one kind sells for 18 GEL, and the other for 25 GEL. The total revenue from the sale of x thousand shirts at 18 GEL each and y thousand at 25 GEL each is given by

$$R(x, y) = 18x + 25y.$$

The company determines that the total cost, in thousands of GEL, of producing x thousand of the 18 GEL shirt and y thousand of the 25 GEL shirt is given by

$$C(x, y) = 4x^2 - 6xy + 3y^2 + 20x + 19y - 12.$$

How many of each type of shirt must be produced and sold in order to maximize profit?



Problem 10.3: City Planning. Four small towns in a rural area wish to pool their resources to build a television station. If the towns are located at the points $(-5, 0)$, $(1, 7)$, $(9, 0)$, and $(0, -8)$ on a rectangular map grid, where units are in miles, at what point $S(a, b)$ should the station be located to minimize the sum of squares of the distances from the towns?

Problem 10.4: Profit under Monopoly. A manufacturer with exclusive rights to a new industrial machine is planning to sell a limited number of them and estimates that if x machines are supplied to the domestic market and y to the foreign market, the machines will sell for $150 - \frac{1}{6}x$ thousand EUR each domestically and for $100 - \frac{1}{20}y$ thousand EUR each abroad. The operating costs are assumed to be constant.

- How many machines should the manufacturer supply to the domestic market to generate the largest possible profit at home?
- How many machines should the manufacturer supply to the foreign market to generate the largest possible profit abroad?
- How many machines should the manufacturer supply to each market to generate the largest possible total profit?
- Is the relationship between the answers in parts a), b), and c) accidental? Explain.

Problem 10.5: The social desirability of an enterprise often involves making a choice between the commercial advantage of the enterprise and the social or ecological loss that may result. For instance, the lumber industry provides paper products to society and income to many workers and entrepreneurs, but the gain may be offset by the destruction of habitable territory for spotted owls and other endangered species. Suppose the social desirability of a particular enterprise is measured by the function

$$D(x, y) = (16 - 6x)x - (y^2 - 4xy + 40),$$

where x measures commercial advantage (profit and jobs) and y measures ecological disadvantage (species displacement, as a percentage) with $x \geq 0$ and $y \geq 0$. The enterprise is deemed desirable if $D \geq 0$ and undesirable if $D < 0$.

- What values of x and y will maximize social desirability? Interpret your result. Is it possible for this enterprise to be desirable?
- The function given in part a) is artificial, but the ideas are not. Research the topic of ethics in industry, and write a paragraph on how you feel these choices should be made.¹

¹Start with the article by K.R. Stollery (1985): Environmental Controls in Extractive Industries, Land Economics, Vol. 61, p. 169.