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Week 12

Lagrange Multipliers

This exercise sheet consists of two parts: at first additional exercises are given the solutions of which are provided with the lecture slides and can serve you as further blueprints when solving similar tasks. Then, the actual homework assignments are stated. Please, hand-in your results of the homework assignments through MSTeams at the date and time specified in MSTeams.

Additional Exercises (see the lecture slides for solutions):

Exercise 12.1: Find the minimum value of the function $f(x,y) = x^2 + y^2$ subject to the constraint $g(x,y) = xy = 1$.

Exercise 12.2: Find the minimum and maximum value of the function $f(x,y) = 8x^2 - 2y$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$.

Exercise 12.3: Determine the extrema of the function $f(x,y) = x^2 - xy + y^2$ subject to the constraint equation $g(x,y) = x^2 + y^2 = 1$.

Exercise 12.4: Determine all relative and absolute extrema of the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$ on the closed (and bounded) unit disk $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Use the Lagrange method to determine the boundary extrema.

Exercise 12.5: Determine the absolute maximum and minimum values of the function $f(x,y) = \mathrm{e}^{x(y+1)}$ on the closed (and bounded) unit disk $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Use the Lagrange method to determine the boundary extrema.

Homework Assignment:

Problem 12.1: Applying the Lagrange method.

a) Find the maximum value of $f(x,y) = xy$ subject to the constraint $g(x,y) = x + y = 1$.

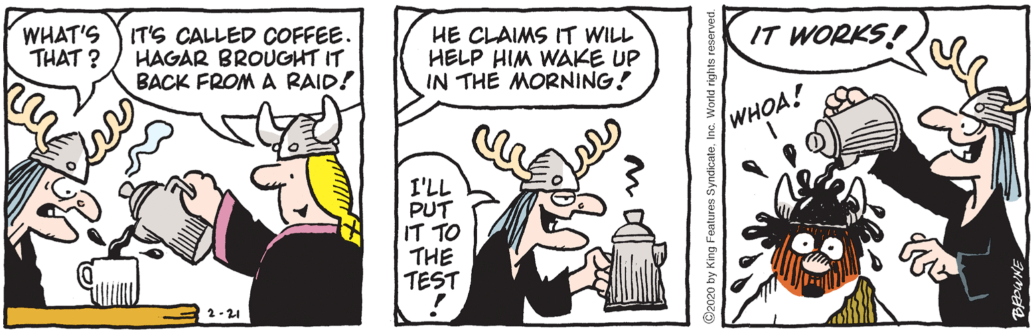
b) Find the maximum and minimum values of $f(x,y) = \mathrm{e}^{xy}$ subject to the constraint $g(x,y) = x^2 + y^2 = 4$.

c) Find the minimum value of the function $f(x,y) = 2x^2 + 4y^2 - 3xy - 2x - 23y + 3$ subject to the constraint $g(x,y) = x + y = 15$.

d) Find the maximum value of the function $f(x,y) = \ln(xy^2)$ subject to $g(x,y) = 2x^2 + 3y^2 = 8$ for $x > 0$ and $y > 0$.

Problem 12.2: A computer company has the Cobb-Douglas production function $P(x,y) = 800x^{3/4}y^{1/4}$ for a certain product, where x is the labor, measured in GEL, and y is the capital, measured in GEL. Suppose that the company can make a total investment in labor and capital of 1000000 GEL. How should it allocate the investment between labor and capital in order to maximize production?

Problem 12.3: A consumer has k GEL to spend on two commodities, the first of which costs a GEL per unit and the second b GEL per unit. Suppose that the utility derived by the consumer from x units of the first commodity and y units of the second commodity is given by the Cobb-Douglas utility function $U(x,y) = x^\alpha y^\beta$, where $0 < \alpha < 1$ and $\alpha + \beta = 1$. Show that utility is maximized when $x = \frac{k\alpha}{a}$ and $y = \frac{k\beta}{b}$.



Problem 12.4: In homework assignment 11.2 we had a specific constraint optimization problem.[1] Solve this applying the method the Lagrange multipliers.

Problem 12.5: Find all interior and boundary extremal points, classify them, and determine the largest and smallest values of the function $f(x, y) = x^4 + 2y^3$ over the given closed, bounded region $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Problem 12.6: Suppose that $p(x, y)$ represents the production of a two-product firm. The company produces x units of the first product at a cost of c_1 each and y units of the second product at a cost of c_2 each. The budget constraint B is a constant given by

$$B(x, y) = c_1 x + c_2 y\,.$$

Use the method of Lagrange multipliers to find the value of λ in terms of p_x, p_y, c_1 and c_2. The resulting equation holds for any production function p and is called the **Law of Equimarginal Productivity**.

[1] A farmer has 300 km^2 on which to plant two crops, celery and lettuce. Each acre of celery costs 250 GEL to plant and tend, and each acre of lettuce costs 300 GEL to plant and tend. The farmer has 81000 GEL available to cover these costs.

a) Suppose the farmer makes a profit of 45 GEL per km^2 of celery and 50 GEL per km^2 of lettuce. Write the profit function, determine how many km^2 of celery and lettuce he should plant to maximize profit, and state the maximum profit. (*Hint*: Since the graph of the profit function is a plane, you will not need to check the interior for possible critical points.)

b) Suppose the farmer's profit function is instead $P(x, y) = -x^2 - y^2 + 600y - 75000$. Assuming the same constraints, how many acres of celery and lettuce should he plant to maximize profit, and what is that maximum profit?